NORTH-HOLLAND

A Weak Majorization Involving the Matrices $A \circ B$ and AB

George Visick
"Briarfield"
40, The Drive
Northwood, Middlesex HA6 1HP, England

Dedicated to M. Fiedler and V. Pták.

Submitted by Iain S. Duff

ABSTRACT

Let A and B be $n \times n$ positive definite matrices, and let the eigenvalues of $A \circ B$ and AB be arranged in decreasing order. Then for all $r > 0$,

$$\sum_{i=k}^{n} \lambda_i^{-r}(AB) \geqslant \sum_{i=k}^{n} \lambda_i^{-r}(A \circ B) \qquad \text{for} \quad k = 1, \dots, n.$$

Furthermore it is shown that

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geqslant \prod_{i=k}^{n} \lambda_i(AB) \qquad \text{for} \quad k = 1, \dots, n$$

is a limiting case of the weak majorization above which settles a conjecture of Bapat and Johnson.

INTRODUCTION

The Hadamard product of two $n \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$ is defined as $A \circ B = (a_{ij} b_{ij})$. Let $G \geqslant 0$ denote that G is positive semidefinite, and $G > 0$ denote that G is strictly positive definite. It is well known by

*Another proof of the Bapat/Johnson conjecture appears in T. Andô: "Majorization Relations for Hadamard Products," on pages 57–64 of this issue.

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Schur's closure theorem (see [2, p. 455]) that the Hadamard product preserves positive (semi)definiteness, i.e., $A \geqslant 0$, $B \geqslant 0$ guarantees $A \circ B \geqslant 0$, while $A > 0$, $B > 0$ guarantees $A \circ B > 0$. Fiedler [4] produced a lower bound for $A \circ B$, viz. $\lambda_n\ (A \circ B) \geqslant \lambda_n(AB^T) \geqslant 0$, where $\lambda_n(\cdot)$ denotes the smallest eigenvalue and B^T is the transpose of B. Later Johnson and Elsner [5] paralleled this result with $\lambda_n(A \circ B) \geqslant \lambda_n(AB) \geqslant 0$. These two inequalities are of special interest in that they relate the Hadamard product to conventional matrix multiplication. On letting $B = A^{-1\,T}$ and $B = A^{-1}$, the inequalities become $A \circ A^{-1\,T} \geqslant I$ and $A \circ A^{-1} \geqslant I$, which are two earlier results of the respective authors above.

Let the eigenvalues of $A \circ B$ and AB be $\lambda_1(A \circ B) \geqslant \lambda_2(A \circ B) \geqslant \cdots \geqslant \lambda_n(A \circ B) \geqslant 0$ and $\lambda_1(AB) \geqslant \lambda_2(AB) \geqslant \cdots \geqslant \lambda_n(AB) \geqslant 0$ respectively. By noting $\lambda_n(A \circ B) \geqslant \lambda_n(AB)$ as above and Oppenheim's inequality (see [2, p. 480])

$$\prod_{i=1}^{n} \lambda_i(A \circ B) = \det(A \circ B) \geqslant \det(AB) = \prod_{i=1}^{n} \lambda_i(AB),$$

it was conjectured in [3, p. 316] and [6] that

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geqslant \prod_{i=k}^{n} \lambda_i(AB) \tag{$*$}$$

for $k = 1, 2, \ldots, n$.

This article confirms the conjecture by proving a more general set of inequalities between the eigenvalues of $A \circ B$ and AB. More specifically, it will be shown that for all $r \geqslant 0$, $A > 0$, $B > 0$,

$$\sum_{i=k}^{n} \lambda_i^{-r}(AB) \geqslant \sum_{i=k}^{n} \lambda_i^{-r}(A \circ B) \qquad (k = 1, \ldots, n)$$

and that $(*)$ is a limiting case of these inequalities.

INTERMEDIATE RESULTS

The main theorem depends on a number of intermediate results, which are mostly given in lemma form. The first three lemmas are stated without proofs but with appropriate references. Lemma 4 is implied in a great deal of literature, but the author was unable to obtain a direct statement. A basic proof based on a technique of Hansen is therefore included. Lemma 5 may be new.

The first three lemmas depend on the concept of majorization. Let $\mathbf{x}$ and $\mathbf{y}$ be real vectors whose elements are such that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$. Then $\mathbf{x}$ is said to weakly majorize $\mathbf{y}$ if

$$\sum_{i=1}^{k} x_i \geqslant \sum_{i=1}^{k} y_i \qquad (k = 1, \dots, n).$$

Should equality occur for $k = n$, then $\mathbf{x}$ is said to majorize $\mathbf{y}$. To link this to matrix theory it is customary to assume that the n eigenvalues of a given $n \times n$ matrix G are in decreasing order, i.e., $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$, and this is assumed throughout this paper. The first lemma is due to Schur (1923), and it could be taken as the starting point of all matrix majorization theory (see [1, p. 218]).

LEMMA 1. *If G is an $n \times n$ Hermitian matrix then*

$$\sum_{i=1}^{k} \lambda_i(I \circ G) \leqslant \sum_{i=1}^{k} \lambda_i(G) \qquad (k = 1, \dots, n) \tag{1}$$

with equality for $k = n$.

This states that the eigenvalues of a Hermitian matrix majorize the diagonal elements of itself. The next lemma relates the eigenvalues of the product of two positive definite matrices to weak majorization (see [1, p. 249]).

LEMMA 2. *Let $A \geqslant 0$, $B \geqslant 0$. Then*

$$\sum_{i=1}^{k} \lambda_i(A)\lambda_{n-i+1}(B) \leqslant \sum_{i=1}^{k} \lambda_i(AB) \leqslant \sum_{i=1}^{k} \lambda_i(A)\lambda_i(B) \qquad (k = 1, \dots, n). \tag{2}$$

The set of inequalities on the right hand side was sharpened by Wang and Gong [7, Theorem 4] by including powers of matrices.

LEMMA 3. *Let $A \geqslant 0$, $B \geqslant 0$. Then for any positive integer m,*

$$\sum_{i=1}^{k} \lambda_i(AB) \leqslant \sum_{i=1}^{k} \lambda_i^{1/m}(A^m B^m) \leqslant \sum_{i=1}^{k} \lambda_i(A)\lambda_i(B)$$

for $k = 1, \ldots, n$.

The first set of inequalities may be rewritten as

$$\sum_{i=1}^{k} \lambda_i(A^{1/m}B^{1/m}) \leqslant \sum_{i=1}^{k} \lambda_i^{1/m}(AB) \qquad (k = 1, \ldots, n). \tag{3}$$

The next lemma's proof relies on the fact that $A \circ B$ is a principal submatrix of the Kronecker product $A \otimes B = (a_{ij}B)$. It therefore follows that there exists a permutation matrix π such that the elements of $A \otimes B$ may be rearranged as

$$\pi^T(A \otimes B)\pi = \begin{bmatrix} A \circ B & X_2 \\ X_2^* & X_3 \end{bmatrix}.$$

LEMMA 4. *For* $A \geqslant 0$, $B \geqslant 0$, *and* $r \in [0, 1]$,

$$(A \circ B)^r \geqslant (A^r \circ B^r). \tag{4}$$

Proof. Following Hansen [8, Lemma 4], set

$$\pi^T(A \otimes B)\pi = \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} \geqslant 0 \qquad \text{with} \quad X_1 = A \circ B.$$

Then for any $\varepsilon > 0$, it is always possible to find a positive constant λ which is large enough to ensure

$$\begin{bmatrix} \varepsilon I & 0 \\ 0 & \lambda I \end{bmatrix} \geqslant \begin{bmatrix} 0 & X_2 \\ X_2^* & X_3 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} A \circ B + \varepsilon I & 0 \\ 0 & \lambda I \end{bmatrix} \geqslant \begin{bmatrix} A \circ B & X_2 \\ X_2^* & X_3 \end{bmatrix} = \pi^T(A \otimes B)\pi.$$

It is also known (see for example [9, p. 131]) that for $F \geqslant 0$, $G \geqslant 0$,

$$F \geqslant G \geqslant 0 \quad \text{implies} \quad F^r \geqslant G^r \geqslant 0 \quad \text{for} \ r \in [0, 1].$$

Applying this to the inequalities above, we have

$$\begin{bmatrix} (A \circ B + \varepsilon I)^r & 0 \\ 0 & \lambda^r I \end{bmatrix} \geqslant \left(\pi^T (A \otimes B) \pi\right)^r$$

$$= \pi^T (A^r \otimes B^r) \pi$$

$$= \begin{bmatrix} A^r \circ B^r & Y_2 \\ Y_2^* & Y_3 \end{bmatrix} \geqslant 0.$$

Simultaneous premultiplication by $[I \ 0]$ and postmultiplication by $[I \ 0]^T$ gives $(A \circ B + \varepsilon I)^r \geqslant A^r \circ B^r$, which produces the desired result on letting $\varepsilon \to 0$. ■

LEMMA 5. *Let* $A > 0$, $C > 0$. *Then*

$$A \circ A^{-1/2} C A^{-1/2} \geqslant (A^{1/2} \circ A^{-1/2}) (I \circ C^{-1})^{-1} (A^{1/2} \circ A^{-1/2}). \quad (5)$$

Proof. It is well known that if $F > 0$ then

$$\begin{bmatrix} F & X \\ X^* & G \end{bmatrix} \geqslant 0 \quad \text{if and only if} \quad G \geqslant X^* F^{-1} X$$

(see [2, p. 472]). Clearly the matrices

$$\begin{bmatrix} I & A^{1/2} \\ A^{1/2} & A \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C^{-1} & A^{-1/2} \\ A^{-1/2} & A^{-1/2} C A^{-1/2} \end{bmatrix}$$

fulfil these conditions and are positive semidefinite. By Schur's closure theorem, their Hadamard product must also be positive semidefinite:

$$\begin{bmatrix} I \circ C^{-1} & A^{1/2} \circ A^{-1/2} \\ A^{1/2} \circ A^{-1/2} & A \circ A^{-1/2} C A^{-1/2} \end{bmatrix} \geqslant 0.$$

This in turn implies the result (5). Note that $C > 0$ implies $I \circ C^{-1} \geqslant 0$. ■

MAIN THEOREM. *If $A > 0$, $B > 0$, then for any positive integer m,*

$$\sum_{i=1}^{k} \lambda_{n-i+1}^{-1/m}(AB) \geqslant \sum_{i=1}^{k} \lambda_{n-i+1}^{-1/m}(A \circ B) \tag{6}$$

for $k = 1, \ldots, n$.

REMARK. By noting that $\lambda_i(AB) = \lambda_i(A^{1/2}BA^{1/2})$, the theorem states that the eigenvalues of $(A^{1/2}BA^{1/2})^{-1/m}$ weakly majorize those of $(A \circ B)^{-1/m}$ for any positive integer m.

Proof.

$$\begin{aligned}(A \circ B)^{1/m} &\geqslant (A^{1/m} \circ B^{1/m}) \qquad \text{by (4) of Lemma 4}\\ &= A^{1/m} \circ A^{-1/2m} C_m A^{-1/2m} \qquad \left(\text{where } C_m = A^{1/2m}B^{1/m}A^{1/2m}\right)\\ &\geqslant (A^{1/2m} \circ A^{-1/2m})\left(I \circ C_m^{-1}\right)^{-1}(A^{1/2m} \circ A^{-1/2m})\end{aligned}$$

by (5) of Lemma 5. It is convenient to restate this as

$$I \circ C_m^{-1} \geqslant (A^{1/2m} \circ A^{-1/2m})(A \circ B)^{-1/m}(A^{1/2m} \circ A^{-1/2m}). \tag{7}$$

Taking eigenvalues of (7) and summing,

$$\sum_{i=1}^{k} \lambda_i\left(I \circ C_m^{-1}\right) \geqslant \sum_{i=1}^{k} \lambda_i\left[(A^{1/2m} \circ A^{-1/2m})^2 (A \circ B)^{1-m}\right]. \tag{8}$$

(This is always true, since $F \geqslant G \geqslant 0$ implies $\lambda_i(F) \geqslant \lambda_i(G)$; see [1, p. 475].) Concentrating on the left hand member of (8),

$$\begin{aligned}\sum_{i=1}^{k} \lambda_i\left(I \circ C_m^{-1}\right) &\leqslant \sum_{i=1}^{k} \lambda_i\left(C_m^{-1}\right) \qquad \text{by (1) of Lemma 1}\\ &= \sum_{i=1}^{k} \lambda_i(A^{-1/m}B^{-1/m})\\ &\leqslant \sum_{i=1}^{k} \lambda_i^{1/m}(A^{-1}B^{-1}) \qquad \text{by (3) of Lemma 3}\\ &= \sum_{i=1}^{k} \lambda_{n-i+1}^{-1/m}(AB)\end{aligned}$$

since $\lambda_i(F^{-1}) = \lambda_{n-i+1}^{-1}(F)$, $k = 1, \dots, n$, if F has n positive eigenvalues. Now concentrating on the right hand member of (8),

$$\sum_{i=1}^{k} \lambda_i\Big(\big(A^{1/2m} \circ A^{-1/2m}\big)^2 (A \circ B)^{-1/m}\Big)$$

$$\geqslant \sum_{i=1}^{k} \lambda_i\Big((A \circ B)^{-1/m}\Big)\lambda_{n-i+1}\Big(\big(A^{1/2m} \circ A^{-1/2m}\big)^2\Big) \qquad \text{by (2), Lemma 2}$$

$$= \sum_{i=1}^{k} \lambda_{n-i+1}^{-1/m}(A \circ B)\lambda_{n-i+1}^{2}(A^{1/2m} \circ A^{-1/2m})$$

$$\geqslant \sum_{i=1}^{k} \lambda_{n-i+1}^{-1/m}(A \circ B), \tag{9}$$

since $A^{1/2m} \circ A^{-1/2m} \geqslant I$ implies $\lambda_i(A^{1/2m} \circ A^{-1/2m}) \geqslant 1$. Combining the above analyses on the left hand and right hand members produces the theorem. ■

Corollary 1. *For $A > 0$, $B > 0$, and $r \geqslant 0$,*

$$\sum_{i=1}^{k} \lambda_{n-i+1}^{-r}(AB) \geqslant \sum_{i=1}^{k} \lambda_{n-i+1}^{-r}(A \circ B) \tag{10}$$

for $k = 1, \dots, n$.

Proof. Let the vectors $\mathbf{x} = (x_i)$, $\mathbf{y} = (y_i)$ contain only positive elements. Then if $\mathbf{x}$ weakly majorizes $\mathbf{y}$, so does $\mathbf{x}_r = (x_i^r)$ weakly majorize $\mathbf{y}_r = (y_i^r)$ for all $r > 1$ (see [1, p. 118]). Applying this to (6) with $r > m^{-1}$, for all positive integers m, achieves (10). ■

Corollary 2. *If $A \geqslant 0$, $B \geqslant 0$, then*

$$\prod_{i=1}^{k} \lambda_{n-i+1}(A \circ B) \geqslant \prod_{i=1}^{k} \lambda_{n-i+1}(AB) \qquad \textit{for} \quad k = 1, \dots, n. \tag{11}$$

Proof. This is the conjecture labeled $(*)$ which was mentioned in the introduction of this paper.

For k positive numbers, $\mathbf{x}(k) = (x_1, x_2, \ldots, x_k)$, the mean power of order r is defined as

$$M_r(\mathbf{x}, k) \equiv \left(\frac{1}{k} \sum_{i=1}^{k} x_i^r \right)^{1/r} \qquad \text{for } r \geqslant 0.$$

Now $M_r(\mathbf{x}, k) \to (\prod_{i=1}^{k} x_i)^{1/k}$ as $r \to 0$ (see [9, p. 15]). Applying this to (10) and raising each side to the power of k achieves (11) for $A > 0,\ B > 0$. Should either A or B be singular, then $\lambda_n(AB) = 0$, so that (11) is actually true for any $A \geqslant 0,\ B \geqslant 0$. ∎

If (10) and (11) are considered extensions of Johnson's inequality $\lambda_n(A \circ B) \geqslant \lambda_n(AB)$, can Fiedler's inequality $\lambda_n(A \circ B) \geqslant \lambda_n(AB^T)$ be extended? By setting C_m to $A^{T\,1/2m} B^{1/m} A^{T\,1/2m}$ in the main theorem and replacing $A^{1/2m} \circ A^{-1/2m}$ by $A^{1/2m} \circ A^{T\,1/2m} \geqslant I$, all the subsequent results follow with $A^T B$ replacing AB. Since (10) and (11) are true for $A^T B$, they are also true for AB^T. This proves

COROLLARY 3.

(i) *If* $A > 0,\ B > 0,\ r \geqslant 0$, *then*

$$\sum_{i=1}^{k} \lambda_{n-i+1}^{-r}(AB^T) \geqslant \sum_{i=1}^{k} \lambda_{n-i+1}^{-r}(A \circ B) \qquad \textit{for} \quad k = 1, \ldots, n. \tag{12}$$

(ii) *If* $A \geqslant 0,\ B \geqslant 0$, *then*

$$\prod_{i=1}^{k} \lambda_{n-i+1}(A \circ B) \geqslant \prod_{i=1}^{k} \lambda_{n-i+1}(AB^T) \qquad \textit{for}\ k = 1, \ldots, n. \tag{13}$$

FURTHER REMARKS

(a)

A correlation matrix is defined as a positive semidefinite matrix with values of unity on the main diagonal (i.e., H is a correlation matrix if $H \geqslant 0$ and $I \circ H = I$). In [10], Bapat and Sunder showed that the identity matrix used in Schur's majorization (1) may be replaced by a correlation matrix H with the majorization still holding. Thus the eigenvalues of G majorize those of $H \circ G$ where G is Hermitian. This result was employed to derive a number of striking inequalities in majorization. In particular, the following set of inequalities gave some motivation to the conjecture $(*)$ (see [3, p. 480] and [6]):

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geqslant \prod_{i=k}^{n} \lambda_i(I \circ A)\lambda_i(B) \geqslant \prod_{i=k}^{n} \lambda_i(A)\lambda_i(B) \tag{14}$$

for $k = 1, \dots, n$ and $A \geqslant 0,\ B \geqslant 0$.

Since for $k = 1, \dots,\ n$ one has $\prod_{i=k}^{n} \lambda_i(AB) \geqslant \prod_{i=k}^{n} \lambda_i(A)\lambda_i(B)$ (see [1, p. 246]), (11) may be extended to

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geqslant \prod_{i=k}^{n} \lambda_i(AB) \geqslant \prod_{i=k}^{n} \lambda_i(A)\lambda_i(B) \tag{15}$$

for $k = 1, \dots, n$.

A natural question to ask is whether the products

$$\prod_{i=k}^{n} \lambda_i(I \circ A)\lambda_i(B) \quad \text{and} \quad \prod_{i=k}^{n} \lambda_i(AB)$$

in (14) and (15) can be ordered in some way. As a counterexample consider $B = A^{-1}$ with A a correlation matrix such that $\lambda_1(A) > 1$. It follows that $\lambda_n(I \circ A)\lambda_n(B) = \lambda_n(A^{-1}) < 1 = \lambda_n(AB)$ while

$$\prod_{i=1}^{n} \lambda_i(I \circ A)\lambda_i(B) = \det(I \circ A) \det B \geqslant \det(AB) = 1,$$

so that the inequality relating $\prod_{i=k}^{n} \lambda_i(I \circ A)\lambda_i(B)$ and $\prod_{i=k}^{n} \lambda_i(AB)$ actually reverses in this case between $k = 1$ and $k = n$.

(b)

There is, however, another expression involving $C = A^{1/2m}B^{1/m}A^{1/2m}$ which may be inserted in (15), since the proof of the main theorem shows

$$\sum_{i=1}^{k} \lambda_{n-i+1}^{-1/m}(AB) \geqslant \sum_{i=1}^{k} \lambda_i\left(I \circ C_m^{-1}\right) \geqslant \sum_{i=1}^{k} \lambda_{n-i+1}^{-1/m}(A \circ B)$$

for $k = 1, \ldots, n$ and positive integer m. Taking the mean power of order $1/m$ and letting $m \to \infty$ produces

$$\prod_{i=1}^{k} \lambda_{n-i+1}(A \circ B) \geqslant \lim_{m \to \infty} \left(\frac{1}{k} \sum_{i=1}^{k} \lambda_i \left(I \circ C_m^{-1} \right) \right)^{-mk} \geqslant \prod_{i=1}^{k} \lambda_i(AB)$$

for $k = 1, \ldots, n$.

As can be seen, the central pivotal member in C_m contains a double inversion. A question that arises is whether a more direct proof could be made by having $A \circ B$ as a function of $I \circ C^{1/m}$, for example. In [11], Ando showed that $I \circ A^2 \geqslant A \circ A$ for $A \geqslant 0$, which is $I \circ C \geqslant A \circ B$ on letting $B = A$. Examples like this, together with Hadamard's extended inequality, $\prod_{i=1}^{k} \lambda_{n-i+1}(I \circ C) \geqslant \prod_{i=1}^{k} \lambda_{n-i+1}(A)B$ for $k = 1, \ldots, n$ (see [1, p. 223]), seem to confound a direct approach to proving the conjecture. The example $I \circ A^2 \geqslant A \circ A$ also shows that $\operatorname{Trace}(A^2) = \operatorname{Trace}(I \circ A^2) \geqslant \operatorname{Trace}(A \circ A)$, so that generally

$$\sum_{i=1}^{k} \lambda_i(A \circ B) \not\geqslant \sum_{i=1}^{k} \lambda_i(AB)$$

and

$$\sum_{i=k}^{n} \lambda_i(A \circ B) \not\geqslant \sum_{i=k}^{n} \lambda_i(AB) \qquad \text{for} \quad k = 1, \ldots, n.$$

(c)

By considering $A \circ B = A \circ A^{-1/2} C A^{-1/2}$ with $C = A^{1/2} B A^{1/2}$, Horn and Johnson (see [3, p. 330]) showed that $\lambda_1(AB) A \circ A^{-1} \geqslant A \circ B \geqslant \lambda_n(AB) A \circ A^{-1}$. Various bounds emerge from (7) to (10) which are slightly different from that of Horn and Johnson. They involve the inverse of C and $(A^{1/2} \circ A^{-1/2})^2$ instead of $A \circ A^{-1}$. They are, however, connected by $A \circ A^{-1} \geqslant (A^{1/2} \circ A^{-1/2})^2$, $\lambda_n(I \circ C^{-1}) \geqslant \lambda_n(A^{-1} B^{-1}) = \lambda_1^{-1}(AB)$, and $\lambda_1(I \circ C^{-1}) \leqslant \lambda_1(A^{-1} B^{-1}) = \lambda_n^{-1}(AB)$. Three new bounds make up the opening results of

Corollary 4. *If $A > 0$, $B > 0$, and $C = A^{1/2} B A^{1/2}$, then*

(*i*) $A \circ B \geqslant \lambda_1^{-1}(I \circ C^{-1})(A^{1/2} \circ A^{-1/2})^2$;
(ii) $I \circ C^{-1} \geqslant \lambda_1^{-1}(A \circ B)(A^{1/2} \circ A^{-1/2})^2$;
(iii) $I \circ C^{-1} \geqslant U(A \circ B)^{-1} U^*$ *for some unitary matrix* U;
(iv) $\operatorname{Trace}(C^{-r}) \geqslant \operatorname{Trace}((A \circ B)^{-r})$ *for* $r \geqslant 0$.

Proof. (i) and (ii) follow immediately from (7) with $m = 1$. (iii) also comes from (7) and is a special case of a more general result in the next corollary. (iv) is simply (10) with $k = n$, and we note that Oppenheim's inequality, $\det(A \circ B) \geqslant \det(AB)$, is a special limiting case of (iv). ∎

The next corollary presents some more inequalities which are closer to majorization theory.

COROLLARY 5

(i) *If* $A > 0$, $B > 0$, *and* $C = A^{1/2}BA^{1/2}$, *then*

$$(I \circ C^{-1})^{1/2} U^*(A \circ B) U (I \circ C^{-1})^{1/2} \geqslant (A^{1/2} \circ A^{-1/2})^2 \geqslant I$$

for some unitary matrix U, *and so*

$$\prod_{i=1}^{k} \lambda_i(I \circ C^{-1}) \lambda_i(A \circ B) \geqslant \prod_{i=1}^{k} \lambda_i^2 (A^{1/2} \circ A^{-1/2})^2 \geqslant 1$$

for $k = 1, \dots, n$.

(ii) *If* $A > 0$, $B > 0$, *and* $r > 0$, *then*

$$\sum_{i=1}^{k} \frac{\lambda_{n-i+1}^r(A \circ B)}{\lambda_{n-i+1}^r(AB)} \geqslant k$$

for $k = 1, \dots, n$.

Proof. In [12], Horn and Mathias show that if

$$\begin{bmatrix} L & X \\ X^* & M \end{bmatrix}$$

is positive semidefinite then there exists a unitary matrix U such that

$$X^*X \leqslant M^{1/2} U^2 L U M^{1/2}.$$

Applying this to (7), which is for $m = 1$

$$\begin{bmatrix} A \circ B & A^{1/2} \circ A^{-1/2} \\ A^{1/2} \circ A^{-1/2} & I \circ C^{-1} \end{bmatrix} \geqslant 0,$$

achieves the first part of (i) above. It also proves (iii) of Corollary 4.

Since $\prod_{i=1}^{k}\sigma_i(FG) \leqslant \prod_{i=1}^{k}\sigma_i(F)\sigma_i(G)$ for $k = 1,\dots,n$ [1, p. 246], where F and G are $n \times n$ matrices and the σ_i's denote the decreasing singular values of a matrix, we have

$$\prod_{i=1}^{k}\lambda_i^2(A^{1/2}\circ A^{-1/2}) \leqslant \prod_{i=1}^{k}\lambda_i((I\circ C^{-1})(U^*(A\circ B)U^*)$$

$$\leqslant \prod_{i=1}^{k}\lambda_i(I\circ C^{-1})\lambda_i(A\circ B) \qquad \text{for} \quad k = 1,\dots,n.$$

This concludes the proof of (i) above.

To prove (ii) we need another result on the mean power of the positive numbers x_i ($i = 1,\dots,k$) as defined in Corollary 2. If for $r > 0$

$$M_r(\mathbf{x},k) \equiv \left(\frac{1}{k}\sum_{i=1}^{k}x_i^r\right)^{1/r},$$

then

$$M_r(\mathbf{x},k) \geqslant \left(\prod_{i=1}^{k}x_i\right)^{1/k}$$

Now (11) may be rewritten as

$$\prod_{i=1}^{k}\frac{\lambda_{n-i+1}(A\circ B)}{\lambda_{n-i+1}(AB)} \geqslant 1 \qquad \text{for} \quad k = 1,\dots,n,$$

so that letting $x_i = \lambda_{n-i+1}(A\circ B)/\lambda_{n-i+1}(AB)$ for $i = 1,\dots,k$ and employing the above property of the mean power proves (ii). ■

CONCLUDING REMARKS

The set of inequalities of Wang and Gong in Lemma 3,

$$\sum_{i=1}^{k}\lambda_i^{1/m}(A^mB^m) \leqslant \sum_{i=1}^{k}\lambda_i(A)\lambda_i(B) \qquad \text{for} \quad k = 1,\dots,n,$$

with integer $m > 0$ may be shown to produce

$$\sum_{i=1}^{k} \lambda_i^r(AB) \leqslant \sum_{i=1}^{k} \lambda_i^r(A)\lambda_i^r(B) \qquad \text{for} \quad k = 1, \dots, n \text{ and } r > 0.$$

Applying the mean power of order r to each side and letting $r \to 0$ yields

$$\prod_{i=1}^{k} \lambda_i(AB) \leqslant \prod_{i=1}^{k} \lambda_i(A)\lambda_i(B) \qquad \text{for} \quad k = 1, \dots, n,$$

which is Horn's result when applied to the product of positive definite matrices.

The set of inequalities in (10) may also be bounded as follows:

$$\sum_{i=k}^{n} \lambda_i^{-r}(A)\lambda_i^{-r}(B) \geqslant \sum_{i=k}^{n} \lambda_i^{-r}(AB) \geqslant \sum_{i=k}^{n} \lambda_i^{-r}(A \circ B) \qquad \text{for} \quad k = 1, \dots, n$$

and $A > 0$, $B > 0$, $r > 0$.

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